ラプラシアンの極座標表示

§A　2次元の場合

直交座標と極座標の変換　$x=R\cos\varphi$　$y=R\sin\varphi$　　$R=\sqrt{x^2+y^2}$　$\varphi=\tan^{-1}(\frac{y}{x})$　　$f(x,y)=f(R\cos\varphi,R\sin\varphi)=\hat{f}(R,\varphi)$

$\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$　を　R,φ　で表わす。連鎖律は　$\frac{\partial}{\partial x}=\frac{\partial R}{\partial x}\frac{\partial}{\partial R}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}$　−(1)　$\frac{\partial}{\partial y}=\frac{\partial R}{\partial y}\frac{\partial}{\partial R}+\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}$　−(2)

●A1　$\partial/\partial x$　を極座標で表す

$$\frac{\partial R}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2)^{\frac{1}{2}}=\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\times 2x=\frac{x}{R}=\frac{R\cos\varphi}{R}=\cos\varphi \quad -(1-1) \qquad \frac{\partial(\tan^{-1}t)}{\partial t}=\frac{1}{1+t^2}\ \text{より}$$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}\tan^{-1}(\frac{y}{x})=\frac{1}{1+(\frac{y}{x})^2}\times\frac{\partial}{\partial x}(\frac{y}{x})=\frac{x^2}{x^2+y^2}\times(\frac{-y}{x^2})=-\frac{y}{x^2+y^2}=-\frac{R\sin\varphi}{R^2}=-\frac{\sin\varphi}{R} \quad -(1-2)$$

(1−1)(1−2)を(1)に代入して　$\frac{\partial}{\partial x}=\cos\varphi\frac{\partial}{\partial R}-\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi}$　−(1−3)

●A2　$\partial^2/\partial x^2$　を極座標で表す

(1−3)より　$\frac{\partial^2}{\partial x^2}\ =\ (\cos\varphi\frac{\partial}{\partial R}-\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi})\times(\cos\varphi\frac{\partial}{\partial R}-\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi})$

$$=\begin{array}{l}\cos\varphi\frac{\partial}{\partial R}(\cos\varphi\frac{\partial}{\partial R})\\ -\cos\varphi\frac{\partial}{\partial R}(\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi})\\ -\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi}(\cos\varphi\frac{\partial}{\partial R})\\ +\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi}(\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi})\end{array}
=\begin{array}{l}\cos^2\varphi\frac{\partial^2}{\partial R^2}\\ -\cos\varphi(-\frac{\sin\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{\sin\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi})\\ -\frac{\sin\varphi}{R}(-\sin\varphi\frac{\partial}{\partial R}+\cos\varphi\frac{\partial^2}{\partial\varphi\partial R})\\ +\frac{\sin\varphi}{R}(\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi}+\frac{\sin\varphi}{R}\frac{\partial^2}{\partial\varphi^2})\end{array}
=\begin{array}{l}\cos^2\varphi\frac{\partial^2}{\partial R^2}\\ +\frac{\cos\varphi\sin\varphi}{R^2}\frac{\partial}{\partial\varphi}-\frac{\cos\varphi\sin\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi}\\ +\frac{\sin^2\varphi}{R}\frac{\partial}{\partial R}-\frac{\sin\varphi\cos\varphi}{R}\frac{\partial^2}{\partial\varphi\partial R}\\ +\frac{\sin\varphi\cos\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{\sin^2\varphi}{R^2}\frac{\partial^2}{\partial\varphi^2}\end{array}$$

$$=\ \cos^2\varphi\frac{\partial^2}{\partial R^2}+\frac{2\cos\varphi\sin\varphi}{R^2}\frac{\partial}{\partial\varphi}-\frac{2\cos\varphi\sin\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi}+\frac{\sin^2\varphi}{R}\frac{\partial}{\partial R}+\frac{\sin^2\varphi}{R^2}\frac{\partial^2}{\partial\varphi^2} \quad -(1-4)$$

●A3　$\partial/\partial y$　を極座標で表す

$$\frac{\partial R}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2)^{\frac{1}{2}}=\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\times 2y=\frac{y}{R}=\frac{R\sin\varphi}{R}=\sin\varphi \quad -(2-1)$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial}{\partial y}\tan^{-1}(\frac{y}{x})=\frac{1}{1+(\frac{y}{x})^2}\times\frac{\partial}{\partial y}(\frac{y}{x})=\frac{x^2}{x^2+y^2}\times(\frac{1}{x})=\frac{x}{x^2+y^2}=\frac{R\cos\varphi}{R^2}=\frac{\cos\varphi}{R} \quad -(2-2)$$

(2−1)(2−2)を(2)に代入して　$\frac{\partial}{\partial y}=\sin\varphi\frac{\partial}{\partial R}+\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi}$　−(2−3)

●A4　$\partial^2/\partial y^2$　を極座標で表す

$$\frac{\partial^2}{\partial y^2}\ =\ (\sin\varphi\frac{\partial}{\partial R}+\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi})\times(\sin\varphi\frac{\partial}{\partial R}+\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi})$$

$$=\begin{array}{l}\sin\varphi\frac{\partial}{\partial R}(\sin\varphi\frac{\partial}{\partial R})\\ +\sin\varphi\frac{\partial}{\partial R}(\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi})\\ +\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi}(\sin\varphi\frac{\partial}{\partial R})\\ +\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi}(\frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi})\end{array}
=\begin{array}{l}\sin^2\varphi\frac{\partial^2}{\partial R^2}\\ +\sin\varphi(-\frac{\cos\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{\cos\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi})\\ +\frac{\cos\varphi}{R}(\cos\varphi\frac{\partial}{\partial R}+\sin\varphi\frac{\partial^2}{\partial\varphi\partial R})\\ +\frac{\cos\varphi}{R}(-\frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi}+\frac{\cos\varphi}{R}\frac{\partial^2}{\partial\varphi^2})\end{array}
=\begin{array}{l}\sin^2\varphi\frac{\partial^2}{\partial R^2}\\ -\frac{\sin\varphi\cos\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{\sin\varphi\cos\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi}\\ +\frac{\cos^2\varphi}{R}\frac{\partial}{\partial R}-\frac{\cos\varphi\sin\varphi}{R}\frac{\partial^2}{\partial\varphi\partial R}\\ -\frac{\cos\varphi\sin\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{\cos^2\varphi}{R^2}\frac{\partial^2}{\partial\varphi^2}\end{array}$$

$$=\ \sin^2\varphi\frac{\partial^2}{\partial R^2}-\frac{2\sin\varphi\cos\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{2\sin\varphi\cos\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi}+\frac{\cos^2\varphi}{R}\frac{\partial}{\partial R}+\frac{\cos^2\varphi}{R^2}\frac{\partial^2}{\partial\varphi^2} \quad -(2-4)$$

●A5 $\partial^2/\partial x^2+\partial^2/\partial y^2$ の極座標表示

(1−4) (2−4)を辺々加える。

$$\frac{\partial^2}{\partial x^2} = \cos^2\varphi\frac{\partial^2}{\partial R^2}+\frac{2\cos\varphi\sin\varphi}{R^2}\frac{\partial}{\partial\varphi}-\frac{2\cos\varphi\sin\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi}+\frac{\sin^2\varphi}{R}\frac{\partial}{\partial R}+\frac{\sin^2\varphi}{R^2}\frac{\partial^2}{\partial\varphi^2} \quad -(1-4)$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\varphi\frac{\partial^2}{\partial R^2}-\frac{2\sin\varphi\cos\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{2\sin\varphi\cos\varphi}{R}\frac{\partial^2}{\partial R\partial\varphi}+\frac{\cos^2\varphi}{R}\frac{\partial}{\partial R}+\frac{\cos^2\varphi}{R^2}\frac{\partial^2}{\partial\varphi^2} \quad -(2-4)$$

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2} \quad -(3)$$

§B 3次元の場合

変換 $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, $z=r\cos\theta$

$r=\sqrt{x^2+y^2+z^2}$, $\varphi=\tan^{-1}(\frac{y}{x})$, $\theta=\tan^{-1}(\frac{\sqrt{x^2+y^2}}{z})$

$$f(x,y,z)=f(r\sin\theta\cos\varphi,r\sin\theta\sin\varphi,r\cos\theta)=\hat{f}(r,\theta,\varphi)$$

$\varDelta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ を r,θ,φ で表示する。

●B1 2次元の結果から

$R=r\sin\theta$ と置くと変換は $x=R\cos\varphi$, $y=R\sin\varphi$, $z=z$ →2 次元の結果から $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}$ (3)

$$\to \varDelta=\frac{\partial^2}{\partial z^2}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=\frac{\partial^2}{\partial z^2}+\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}\right)=\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial R^2}\right)+\left(\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}\right) \quad (4)$$

次に $z=r\cos\theta$, $R=r\sin\theta$ の変換をすると $r=\sqrt{z^2+R^2}$, $\theta=\tan^{-1}(\frac{R}{z})$ 2 次元の場合を $R\to r$, $x\to z$, $y\to R$, $\varphi\to\theta$ で置き換えることにより

$\frac{\partial R}{\partial y}=\sin\varphi$ (2−1) → $\frac{\partial r}{\partial R}=\sin\theta$ (4−1) $\frac{\partial\varphi}{\partial y}=\frac{\cos\varphi}{R}$ (2−2) → $\frac{\partial\theta}{\partial R}=\frac{\cos\theta}{r}$ (4−2)

$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}$ (3) → $\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial R^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ (4−3)

●B2 ラプラシアンの極座標表示

また $\frac{1}{R}\frac{\partial}{\partial R}=\frac{1}{R}(\frac{\partial r}{\partial R}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial R}\frac{\partial}{\partial\theta})=\frac{1}{r\sin\theta}(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta})=\frac{1}{r}\frac{\partial}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}$ (4−4)

(4)(4−1)(4−2)(4−3)(4−4)以上より

$$\Delta = \frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2} = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$= \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \quad (4-5)$$

さらに $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})=\frac{1}{\sin\theta}(\cos\theta\frac{\partial}{\partial\theta}+\sin\theta\frac{\partial^2}{\partial\theta^2})=\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}$ を(4−5)に代入すると

$$\Delta = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \quad (4-6)$$